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Analytic Properties of Current-Algebra Vertex Functions*

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An expansion of the vector-current vertex is obtained in terms of the Joos spinor amplitudes. As these are free of kinematic singularities, one can remove them from the vertex that enters into the algebra-of-currents sum rules. If one assumes unsubtracted dispersion relations for the resulting form factors, then the sum rules can be cast into a form which involves Bessel transforms of these form factors.

I. INTRODUCTION

RECENTLY there has been an extensive study of the kinematic singularities of scattering amplitudes involving particles of arbitrary spin.[1,2] The simpler problem of the singularity structure of current vertex functions has been neglected. The interest in current algebra and attempts to saturate it may require the knowledge of the analytic properties of these functions. In this article, we obtain a parametrization of the vector vertex function in such a way that all kinematic singularities are explicitly exhibited. Using this parametrization, a proof is obtained of the fact that the particular combination of vertex functions appearing in the expectation value of the current taken between states of infinite momentum has, aside from a trivial threshold factor, no other kinematic singularities. It is this combination that enters into the algebra of currents. Although the discussion will be carried through for vector currents, only slight modifications would be necessary for the discussion of other currents.

Most of the discussion of the parametrization follows that given by Cohen-Tannoudji, Morel, and Navelet[2] for scattering amplitudes. In outline, the procedure consists of expanding the vertex function in a generalization of the Joos[3] spinor amplitudes. One would then have to show that these amplitudes are free of kinematic singularities. For the scattering case, this was done by Williams[4] through an extension of the Hall-Wightman theorem.[5] For a scalar vertex, the proof would be a simpler version of that given by Williams. For the vector case, or in general for higher tensor currents, a trivial extension of the above theorem would be required. The details of this extension have not been carried out.

This expansion and the investigation of the singularities of the vertex entering into the current-algebra sum rules is done in Secs. II, III, and IV, where Sec. III is devoted to a recapitulation of the results of current algebra. If one assumes that the relevant kinematical singularity-free amplitude satisfied an unsubtracted dispersion relation, then an interesting form for the sum rules may be obtained and is presented in Sec. V. In the Appendix, the properties of a scalar vertex are discussed.

II. EXPANSION OF THE VECTOR-CURRENT VERTEX

The vertex function for a vector current is defined as

$$\Gamma_{\lambda_1,\lambda_2}{}^{\mu}(p_1,p_2)=(2\pi)^3(4\omega_1\omega_2)^{1/2}\langle\mathbf{p}_1,\lambda_1|J^{\mu}(0)|\mathbf{p}_2,\lambda_2\rangle. \quad (2.1)$$

In the above, J^μ is any current transforming as a four-vector, $\omega_i=(\mathbf{p}_i^2+m_i^2)^{1/2}$, and the λ refer to the spin of the state. We do not specify the spin structure of the states yet. For much of the subsequent discussion any prescription may be used, i.e., $|\mathbf{p},\lambda\rangle$ may be a helicity state,[6] Wigner spin state,[7] or the states introduced by Bardakci and Segrè.[8] Under a Lorentz transformation Λ, the vertex function, behaves as follows:

$$D_{\lambda_1,\eta_1}{}^{(j_1)}(W^{-1}(\Lambda,p_1))\times\Gamma_{\eta_1,\eta_2}{}^{\mu}(\Lambda p_1,\Lambda p_2)D_{\eta_2,\lambda_2}{}^{(j_2)}(W(\Lambda,p_2))=\Lambda^{\mu}{}_{\nu}\Gamma_{\lambda_1,\lambda_2}{}^{\nu}(p_1,p_2), \quad (2.2)$$

where $D^{(j)}(W(\Lambda,p))$ is a spin-j representation of a Wigner rotation determined by Λ and p. If $L(p)$ is a Lorentz transformation which takes a state at rest to one moving with momentum $\mathbf{p}$ according to some set prescription, i.e.,

$$|\mathbf{p},\lambda\rangle=(2m/2\omega)^{1/2}U(L(p))|\mathbf{o},\lambda\rangle, \quad (2.3)$$

then the Wigner rotation is given by

$$D^{(j)}(W(\Lambda,p))=D^{(j,0)}(L^{-1}(\Lambda p)\Lambda L(p)), \quad (2.4)$$

with $D^{(j,0)}$ denoting the $(j,0)$ representation of the Lorentz group.

The spinor ket state, $|\mathbf{p},A\rangle$,[9] is obtained by

$$|\mathbf{p},A\rangle=|\mathbf{p},\lambda\rangle D_{\lambda,A}{}^{(j,0)}(L^{-1}(p)). \quad (2.5)$$

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† Alfred P. Sloan Foundation Fellow.

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[2] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N.Y.) 46, 239 (1968).
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[5] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 31, No. 5 (1957).
[6] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).
[7] E. Wigner, Ann. Math. 40, 149 (1939).
[8] K. Bardakci and G. Segrè, Phys. Rev. 159, 1263 (1967).
[9] Greek subscripts indicate helicity or the z component of spin in the rest frame. Capital subscripts indicate generalized spinor indices.

The nonunitarity of the $(j,0)$ representations of the Lorentz group requires that bra spinor states, which will transform under the same representations of the Lorentz group, be defined as

$$\langle p,\dot{A}| = \langle p,\lambda|[C^{-1}D^{(j,0)}(L^{-1}(p))]_{\lambda,\dot{A}}. \quad (2.6)$$

In the above, C is the matrix which transforms the spin-j representation of the rotation group into its complex conjugate;

$$C_{\lambda,\eta}{}^{(j)} = (-1)^{j-\eta}\delta_{\lambda,-\eta}.$$

The vertex function taken between spinor states, $\Gamma_{\dot{A}_1,A_2}{}^{\mu}$, may be related to the one used previously by

$$\Gamma_{\lambda_1,\lambda_2}{}^{\mu} = [D^{-1(j_1,0)}(L^{-1}(p_1))]_{\lambda_1,\dot{A}} \times\Gamma_{\dot{A}_1,A_2}{}^{\mu}D_{A_2,\lambda_2}{}^{(j_2,0)}(L^{-1}(p_2)), \quad (2.7)$$

which has simpler Lorentz transformation properties, i.e.,

$$D_{\dot{A}_1,\dot{B}_1}{}^{(j_1,0)}(\Lambda)\Gamma_{\dot{B}_1,B_2}{}^{\mu}(\Lambda p_1,\Lambda p_2)D_{\dot{B}_1,A_2}{}^{(j_2,0)}(\Lambda) = \Lambda^{\mu}{}_{\nu}\Gamma_{\dot{A}_1,A_2}{}^{\nu}(p_1,p_2). \quad (2.8)$$

Using Clebsch-Gordan coefficients, we form

$$\Gamma_{J,M}{}^{\mu} = \sum_{\dot{A}_1,A_2}\langle J,M|j_1,\dot{A}_1;j_2,A_2\rangle\Gamma_{\dot{A}_1,A_2}{}^{\mu}. \quad (2.9)$$

The transformation property of $\Gamma_{J,M}{}^{\mu}$ is

$$D_{M',M}{}^{(J,0)}(\Lambda)\Gamma_{J,M'}{}^{\mu}(\Lambda p_1,\Lambda p_2) = \Lambda^{\mu}{}_{\nu}\Gamma_{J,M}{}^{\nu}(p_1,p_2). \quad (2.10)$$

Analogous to the expansion of the scattering amplitude in Refs. 3 and 4, we expand $\Gamma_{J,M}{}^{\mu}$ as

$$\begin{aligned}\Gamma_{J,M}{}^{\mu}(p_1,p_2) &= [A_J{}^{(1)}P^{\mu}+B_J{}^{(1)}Q^{\mu}]Y_{J,M}(\mathbf{e}) \\ &\quad+\langle J,M|1,\alpha;J-1,\beta\rangle Y_{J-1,\beta}(\mathbf{e})[C_J{}^{(1)}\rho_{\alpha}{}^{\mu}(p_1) \\ &\quad+iD_J{}^{(1)}\epsilon^{\mu}{}_{\nu\lambda\sigma}\rho_{\alpha}{}^{\nu}p_1{}^{\lambda}p_2{}^{\sigma}]. \quad (2.11)\end{aligned}$$

In the equation above, $P=p_1+p_2$, $Q=p_2-p_1$; $\mathbf{e}$ is a vector formed from p_1 and p_2 by $\mathbf{e}=\omega_1\mathbf{p}_2-\omega_2\mathbf{p}_1-i\mathbf{p}_1\times\mathbf{p}_2$ and is such that when p_1 and p_2 are subjected to a Lorentz transformation Λ, $Y_{JM}(e)$ goes to $D_{M',M}{}^{(J,0)}(\Lambda)\times Y_{J,M'}(\mathbf{e})$. $\rho_{\alpha}{}^{\mu}$ transforms under the (1,0) Lorentz transformation in the index α and is given by $\rho_{\alpha}{}^{\mu}(p) = \langle 1,\alpha|\tfrac{1}{2},\beta;\tfrac{1}{2},\gamma\rangle(i\sigma_y\sigma^{\mu}\sigma\cdot p)_{\beta\gamma}$, with $\sigma^{\mu}=(1,\boldsymbol{\sigma})$. A_J, B_J, C_J, and D_J are functions of the Lorentz invariants formed from p_1 and p_2 and *are free of kinematic singularities*.[10]

At this point let us choose a definite Lorentz frame, namely, the Breit coordinate system,[11] where

$$\begin{aligned}P^{\mu} &= ((2m_1{}^2+2m_2{}^2+q^2)^{1/2};0;0;0), \\ Q^{\mu} &= ((m_2{}^2-m_1{}^2)/(2m_1{}^2,+2m_2{}^2+q^2)^{1/2}; \\ &\qquad T/(2m_1{}^2+2m_2{}^2+q^2)^{1/2};0;0),\end{aligned} \quad (2.12)$$

with $T=[q^4+2q^2(m_1{}^2+m_2{}^2)+(m_1{}^2-m_2{}^2)^2]^{1/2}$. In this coordinate system, $\mathbf{e}$ points along the z direction with magnitude T, and $Y_{JM}(\mathbf{e})$ is proportional to $T^J\delta_{M,0}$. Absorbing some common factors into the $A_J{}^{(1)}$–$D_J{}^{(1)}$, we obtain

$$\begin{aligned}\Gamma_{J,M}{}^{0}(q^2) &= T^J\{(2m_1{}^2+2m_2{}^2+q^2)^{1/2}A_J(q^2) \\ &\quad+[(m_2{}^2-m_1{}^2)/(2m_1{}^2+2m_2{}^2+q^2)^{1/2}]B_J(q^2) \\ &\quad-J^{1/2}C_J(q^2)/(2m_1{}^2+2m_2{}^2+q^2)^{1/2}\}\delta_{M,0}, \\ \Gamma_{J,M}{}^{z}(q^2) &= [T^{J-1}/(2m_1{}^2+2m_2{}^2+q^2)^{1/2}] \\ &\quad\times[T^2B_J(q^2)+J^{1/2}(q^2+3m_1{}^2+m_2{}^2)C_J(q^2)]\delta_{M,0}, \\ \Gamma_{J,M}{}^{-x+iy}(q^2) &= -(J+1)^{1/2}T^{J-1} \\ &\quad\times[(q^2+3m_1{}^2+m_2{}^2+T)/(2m_1{}^2+3m_1{}^2+q^2)^{1/2}] \\ &\quad\times[C_J(q^2)+TD_J(q^2)]\delta_{M,-1}, \\ \Gamma_{J,M}{}^{x+iy}(q^2) &= (J+1)^{1/2}T^{J-1} \\ &\quad\times[(q^2+3m_1{}^2+m_2{}^2-T)/(2m_1{}^2+2m_2{}^2+q^2)^{1/2}] \\ &\quad\times[C_J(q^2)-TD_J(q^2)]\delta_{M,1}.\end{aligned} \quad (2.13)$$

At this point, we make a definite choice for $L(p)$. As both $\mathbf{p}_1$ and $\mathbf{p}_2$ are along the x axis, choosing $L(p)$ to be a pure boost will give us a vertex function for either the Wigner spin states[7] or for the states introduced by Bardakci and Segrè.[8] The vertex function for helicity states[6] may be obtained by letting $\lambda_1\to-\lambda_1$ and multiplying the result by $(-1)^{j_1-\lambda_1}$. With this pure boost choice for $L(p)$, we find,

$$D_{\lambda,A}{}^{(j,0)}(L(p)) = \delta_{\lambda A}((\omega-p)/m)^{\lambda}. \quad (2.14)$$

Using explicitly the momenta in the Breit frame, we obtain[12]

$$\Gamma_{\lambda_1,\lambda_2}{}^{\mu}(q^2) = (-1)^{j_1-\lambda_1}\left(\frac{m_1{}^2+m_2{}^2+q^2+T}{2m_1m_2}\right)^{\lambda_2} \times(2m_1{}^2+2m_2{}^2+q^2)^{-1/2}\sum_J X_{J;\lambda_1\lambda_2}{}^{\mu}, \quad (2.15\text{a})$$

with

$$\begin{aligned}X_{J;\lambda_1,\lambda_2}{}^{0} &= \langle j_1,-\lambda_1;j_2,\lambda_2|J,0\rangle T^J \\ &\quad\times[(2m_1{}^2+2m_2{}^2+q^2)A_J(q^2) \\ &\quad+(m_2{}^2-m_1{}^2)B_J(q^2)-J^{1/2}C_J(q^2)], \\ X_{J;\lambda_1,\lambda_2}{}^{z} &= \langle j_1,-\lambda_1;j_2,\lambda_2|J,0\rangle T^{J-1} \\ &\quad\times[T^2B_J(q^2)+J^{1/2}(3m_1{}^2+m_2{}^2+q^2)C_J(q^2)], \\ X_{J;\lambda_1,\lambda_2}{}^{-x+iy} &= -(J+1)^{1/2}m_1\langle j_1,-\lambda_1;j_2,\lambda_2|J,-1\rangle T^{J-1} \\ &\quad\times[C_J(q^2)+TD_J(q^2)], \\ X_{J;\lambda_1,\lambda_2}{}^{x+iy} &= (J+1)^{1/2}m_1\langle j_1,-\lambda_1;j_2,\lambda_2|J,1\rangle T^{J-1} \\ &\quad\times[C_J(q^2)-TD_J(q^2)]. \quad (2.15\text{b})\end{aligned}$$

As A_J, B_J, C_J, and D_J have only dynamical singularities in q^2, we may determine all the kinematic singularities of the vertex functions by inspecting the expansion coefficients.

[10] We do not really know where all the singularities of these functions lie, except in cases where the external masses are low. In general, there will occur anomalous thresholds which we consider to be dynamic in origin.

[11] L. Durand, D. DeCelles, and R. Marr, Phys. Rev. **126**, 1882 (1962).

[12] The asymmetry in indices 1 and 2 is due to the fact that p_1 has been given a preferential role in the expansion. Had we chosen p_2, we would get an equally acceptable set of invariant $\bar{A}_J\cdots\bar{D}_J$ related to $A_J\cdots D_J$ by a nonsingular matrix.

III. CURRENT-ALGEBRA SUM RULES

In this section, we shall state the sum rules of current algebra in terms of the current vertex functions. These results have been previously obtained elsewhere[13] and in a very transparent form recently by Bardakci and Segrè.[8]

Let

$$H_{\lambda_1,\lambda_2}(q^2)=d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)\Gamma_{\eta_1,\eta_2}(q^2)d_{\eta_2,\lambda_2}{}^{(j_2)}(\psi_2)\,, \quad (3.1)$$

with

$$\Gamma_{\lambda_1,\lambda_2}(q^2)=(2m_1^2+2m_2^2+q^2)^{1/2}\times\left[\Gamma_{\lambda_1,\lambda_2}{}^0(q^2)-\frac{m_2^2-m_1^2}{T}\Gamma_{\lambda_1,\lambda_2}{}^z(q^2)-q\frac{(2m_1^2+2m_2^2+q^2)^{1/2}}{T}\Gamma_{\lambda_1,\lambda_2}{}^x(q^2)\right]; \quad (3.2)$$

the rotation angles ψ_1 and ψ_2 are determined by

$$\begin{aligned}\cos\psi_1&=(m_2^2-m_1^2+q^2)/T, & \sin\psi_1&=-2m_1q/T\,,\\ \cos\psi_2&=(m_2^2-m_1^2-q^2)/T, & \sin\psi_2&=-2m_2q/T\,.\end{aligned} \quad (3.3)$$

In the above, we use the Wigner spin states. With these definitions behind us, the general result from current algebra, obtained by commuting two time-components of currents and sandwiching the result between states of infinite momentum in the z direction, is

$$\sum_{n,\lambda_n}\{H_{\lambda_i,\lambda_n}{}^\alpha(i-n;q_1^2)H_{\lambda_n,\lambda_f}{}^\beta(n-f;q_2^2)e^{i(\lambda_1+\lambda_2-2\lambda_n)\varphi/2}-H_{\lambda_i,\lambda_n}{}^\beta(i-n;q_2^2)H_{\lambda_n,\lambda_f}{}^\alpha(n-f;q_1^2)e^{-i(\lambda_1+\lambda_2-2\lambda_n)\varphi/2}\}=if^{\alpha\beta\gamma}H_{\lambda_i,\lambda_f}{}^\gamma(i-f;q_{12}^2)e^{-i(\lambda_i-\lambda_f)X}. \quad (3.4)$$

In the above, $H_{\lambda_m,\lambda_n}{}^\alpha(m-n;q^2)$ refers to the particular combination of current vertex function appearing in (3.1) and (3.2) taken between state m and state n, α designates a $U(3)\otimes U(3)$ index, and $f^{\alpha\beta\gamma}$ is a structure constant for this group. The spin designations are those discussed in the previous section [following Eq. (2.13)]. Likewise,

$$\begin{aligned}q_{12}&=(q_1^2+q_2^2+2q_1q_2\cos\varphi)^{1/2}\,,\\ e^{iX}&=(q_1e^{i\varphi/2}+q_2e^{-i\varphi/2})/q_{12}\,.\end{aligned} \quad (3.5)$$

φ is an angle that varies between 0 and 2π. In the next section, we obtain the singularities of the H functions appearing in Eq. (2.4).

IV. KINEMATIC SINGULARITIES OF $H_{\lambda_1,\lambda_2}(q^2)$

The function $H_{\lambda_1,\lambda_2}(q^2)$ appearing in the current-algebra sum rules, Eq. (3.4), has fairly simple analytic properties. We shall show in this section that $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$ has no kinematic singularities and thus has a chance of satisfying a simple dispersion relation in the variable q^2. In terms of the spinor amplitudes introduced in Eq. (2.15), the combination of vertex functions in Eq. (3.2) is

$$\begin{aligned}\Gamma_{\lambda_1,\lambda_2}(q^2)=&(-1)^{j_1-\lambda_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{\lambda_2}\\ &\times\sum_J T^{J-2}\{\langle J,0|j_1,-\lambda_1;j_2,\lambda_2\rangle\\ &\times[A_JT^2+J^{1/2}C_J(m_1^2-m_2^2-q^2)]\\ &+qm_1(J+1)^{1/2}[\langle J,-1|j_1,\lambda_1;j_2,\lambda_2\rangle(C_J+TD_J)\\ &-\langle J,1|j_1,-\lambda_1;j_2,\lambda_2\rangle(C_J-TD_J)]\}\,.\end{aligned} \quad (4.1)$$

Possible singularities may occur at $q^2=0$ and at $q^2=-(m_1\pm m_2)^2$; the last two are roots of T. Let us divide the negative q^2 axis into three parts:

$$\begin{aligned}&\text{I},\ 0>q^2>-(m_1-m_2)^2\,;\\ &\text{II},\ -(m_1-m_2)^2>q^2>-(m_1+m_2)^2\,;\\ &\text{III},\ -(m_1+m_2)^2>q^2>-\infty\,.\end{aligned} \quad (4.2)$$

Using the reflection property of Clebsch-Gordan coefficients,[14]

$$\langle J,M|j_1,\lambda_1;j_2,\lambda_2\rangle=(-1)^{J+j_1+j_2}\langle J,-M|j_1,-\lambda_1;j_2,-\lambda_2\rangle, \quad (4.3)$$

and noting that

$$\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^\lambda=\left(\frac{[(m_1+m_2)^2+q^2]^{1/2}+[(m_1-m_2)^2+q^2]^{1/2}}{2(m_1m_2)^{1/2}}\right)^{2\lambda}=\left(\frac{[(m_1+m_2)^2+q^2]^{1/2}-[(m_1-m_2)^2+q^2]^{1/2}}{2(m_1m_2)^{1/2}}\right)^{-2\lambda}. \quad (4.4)$$

We may obtain the changes in $\Gamma_{\lambda_1,\lambda_2}(q^2)$ and the rotation matrices appearing in Eqs. (3.2) and (3.3) upon going through the cuts defined in (4.2):

$$\begin{aligned}\Gamma_{\lambda_1,\lambda_2}(q^2)&\xrightarrow{\text{I}}(-1)^{\lambda_1-\lambda_2}\Gamma_{\lambda_1,\lambda_2}(q^2)\,,\\ \Gamma_{\lambda_1,\lambda_2}(q^2)&\xrightarrow{\text{II}}(-1)^{j_1+j_2}(-1)^{2\lambda_1}\Gamma_{-\lambda_1,-\lambda_2}(q^2)\,,\\ \Gamma_{\lambda_1,\lambda_2}(q^2)&\xrightarrow{\text{III}}(-1)^{\lambda_1+\lambda_2}\Gamma_{\lambda_1,\lambda_2}(q^2)\,,\end{aligned} \quad (4.5)$$

and

$$\begin{aligned}\psi_{1,2}&\xrightarrow{\text{I}}-\psi_{1,2}\,,\\ \psi_{1,2}&\xrightarrow{\text{II}}\pi-\psi_{1,2}\,,\\ \psi_{1,2}&\xrightarrow{\text{III}}2\pi-\psi_{1,2}\,.\end{aligned} \quad (4.6)$$

The superscripts above the arrows indicate which por-

[13] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966).

[14] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957), p. 42.

tion of the cut along the negative q^2 is traversed. Combining the two preceding relations and using standard properties of the rotation matrices yields

$$H_{\lambda_1,\lambda_2}(q^2) \xrightarrow{\text{I,II,III}} (-1)^{\lambda_1-\lambda_2} H_{\lambda_1,\lambda_2}(q^2). \tag{4.7}$$

Thus, $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$ is free of branch-point singularities and it remains to show that it has no pole singularities.

Poles could occur at the same places where branch points were possible, namely, at $q^2=0$ and

$$q^2=-(m_1\pm m_2)^2.$$

For $q^2\to 0$, $d_{\lambda,\eta}(\psi_{1,2})\sim q^{|\lambda-\eta|}$ and each term in the definition H_{λ_1,λ_2} behaves as $q^{|\lambda_1-\eta|}q^{|\eta-\lambda_2|}$ or $qq^{|\lambda_1-\eta|}\times q^{|(\eta\pm1)-\lambda_2|}$ which in all cases goes to zero at least as fast as $q^{|\lambda_1-\lambda_2|}$, thus canceling the possible pole in $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$.

To eliminate any difficulty at $q^2=-(m_1\pm m_2)^2$ we examine the following functions:

$$E_{\lambda_1,\lambda_2}{}^{(J)}(q^2)=d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)\left[(-1)^{j_1-\eta_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{1/2} T^J\langle J,0|j_1,-\eta_1;j_2,\eta_2\rangle\right]d_{\eta_2\lambda_2}{}^{(j_2)}(\psi_2);$$

$$\begin{aligned}F_{\lambda_1,\lambda_2}{}^{(J)}(q^2)=d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)\bigg[(-1)^{j_1-\eta_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{1/2} T^{J-2}[J^{1/2}(m_1^2+m_2^2-q^2)\langle J,0|j_1,-\eta_1;j_2,\eta_2\rangle\\ +qm_1(J+1)^{1/2}\langle J_2,-1|j_1,-\eta_1;j_2,\eta_2\rangle-qm_1(J+1)^{1/2}\langle J_1,+1|j_1,-\eta_1;j_2,\eta_2\rangle]\bigg]d_{\eta_2,\lambda_2}(j_2)(\psi_2);\end{aligned} \tag{4.8}$$

$$\begin{aligned}G_{\lambda_1,\lambda_2}{}^{(J)}(q^2)=d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)\bigg[(-1)^{j_1-\eta_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{\eta_2}\\ \times qm_1(J+1)^{1/2}T^{J-1}(\langle J,-1|j_1,-\eta_1;j_2,\eta_2\rangle+\langle J,1|j_1,-\eta_1;j_2,\eta_2\rangle)\bigg]d_{\eta_2,\lambda_2}{}^{(j_2)}(\psi_2).\end{aligned}$$

If it can be shown that these functions have no poles, then neither does $H_{\lambda_1,\lambda_2}(q^2)$, since

$$H_{\lambda_1,\lambda_2}(q^2)=\sum_J(E_{\lambda_1,\lambda_2}{}^{(J)}A_J+F_{\lambda_1,\lambda_2}{}^{(J)}C_{(J)}+G_{\lambda_1,\lambda_2}{}^{(J)}D_J). \tag{4.9}$$

The detail of the proof will be carried out for $q^2=-(m_1-m_2)^2$; the discussion for $q=+(m_1+m_2)^2$ goes through with minor modification. Let $\psi_\pm=\frac{1}{2}(\psi_1+\psi_2)$. The singularities in $\psi_{1,2}$ now separate as may be seen from

$$e^{i\psi_+}=(m_1-m_2+iq)/[(m_1-m_2)^2+q^2]^{1/2};$$
$$e^{i\psi}=(m_1+m_2+iq)/[(m_1+m_2)^2+q^2]^{1/2}. \tag{4.10}$$

Likewise, $(m_1^2+m_2^2+q^2+T)/2m_1m_2$ goes to 1 as q^2 approaches $-(m_1-m_2)^2$ and

$$\begin{aligned}&d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)(-1)^{j-\eta_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{\eta_2}\\ &\times\langle J,M|j_1,-\eta_1;j_2,\eta_2\rangle d_{\eta_2,\lambda_2}{}^{(j_2)}(\psi_2)\to\\ &\times d_{\lambda_1,\xi_1}{}^{-1(j_1)}(X_+)d_{\xi_1,\eta_1}{}^{-1(j_1)}(\psi_-)(-1)^{j_1-\eta_1}\\ &\times\langle J,M|j_1,-\eta_1;j_2,\eta_2\rangle d_{\eta_2,\xi_2}{}^{(j_2)}(-\psi_-)d_{\xi_2,\lambda_2}{}^{(j_2)}(X_+),\end{aligned} \tag{4.11}$$

with $X_+=\lim\psi_+$. Using identities relating Clebsch-Gordan coefficients and the rotation matrices,[15] we obtain, as the limit of Eq. (2.11),

$$d_{\lambda_1,\xi_1}{}^{-1(j_1)}(X_+)\langle J,M'|j_1,\xi_1;j_2\xi_2\rangle d_{\xi_2,\lambda_2}{}^{(j_2)}\times(X_+)d_{M',M}{}^{(J)}(\psi_-). \tag{4.12}$$

We thus note that in the limit of $q^2\to-(m_1-m_2)^2$, E, F, and G approach a combination of $d_{M'0}{}^J(\psi_-)$ and $d_{M',\pm1}{}^{(J)}(\psi_-)$ which may be related to associated Legendre functions[16] and we may use asymptotic expansions of this function to show that E, F, and G have no poles at $q^2=-(m_1-m_2)^2$. For example, $E^{(J)}$ is proportional to $T^Jd_{M',0}{}^{(J)}(\psi_-)=[(J-M)!/(J+M)!]^{1/2}\times T^JP_M{}^J(\psi_-)$, which has no pole at $\psi_-=\infty$. The proof for $F^{(J)}$ and $G^{(J)}$ goes through similarly. To obtain the result for $G^{(J)}$, we use the fact that

$$d_{M,1}{}^{(J)}(\psi_-)-d_{M,-1}{}^{(J)}(\psi_-)=\left(\frac{(J-M)!}{(J+M)!}\right)^{1/2}\frac{2M}{[J(J+1)]^{1/2}\sin\psi_-}P_M{}^J(\psi_-), \tag{4.13}$$

which likewise has no pole upon multiplication by T^{J-1}. For $F^{(J)}$, the algebra is somewhat more involved but just as direct. Thus we have proven that $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$ has no kinematic singularities.

It is tempting to assume that $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$ satis-

[15] Reference 14, p. 61.

[16] Reference 14, p. 59 and recursion relation on p. 61.

fies an unsubstracted dispersion relation[17] in q^2:

$$H_{\lambda_1,\lambda_2}(q^2)=\frac{q^{|\lambda_1-\lambda_2|}}{\pi}\int\frac{dq^2}{q^2+\sigma^2}\,\mathrm{Im}\left(\frac{H_{\lambda_1,\lambda_2}(\sigma^2)}{\sigma^{|\lambda_1-\lambda_2|}}\right),\quad(4.14)$$

with the integration running along the dynamical singularities of H. Some consequences of this assumption are presented in the next section.

V. SUM RULES FOR BESSEL TRANSFORMS OF VERTEX FUNCTIONS

The structure of the sum rules in Eq. (3.6) suggests that we consider Bessel transforms of the H_{λ_1,λ_2}'s. Using a standard identify[18] for Bessel functions,

$$\left(\frac{q_1+q_2e^{i\varphi}}{(q_1^2+q_2^2+2q_1q_2\cos\varphi)^{1/2}}\right)^n J_n(b(q_1^2+q_2^2+2q_1q_2\cos\varphi)^{1/2})=\sum_{k=\infty}^{\infty}J_k(bq_2)J_{n-k}(bq_1)e^{ik\varphi},\quad(5.1)$$

and Eq. (3.4), we obtain the sum rule

$$\sum_n\left[L_{\lambda_i;\lambda_f-k}{}^{\alpha}(i-n;b_1)L_{\lambda_f-k,\lambda_f}{}^{\beta}(n-f;b_2)-L_{\lambda_i,\lambda_i+k}{}^{\beta}(i-n,b_2)L_{\lambda_i+k,\lambda_f}{}^{\alpha}(n-f,b_1)\right]=if^{\alpha\beta\gamma}\frac{\delta(b_1-b_2)}{b_1}L_{\lambda_i,\lambda_f}{}^{\gamma}(i-f;b_1),\quad(5.2)$$

where

$$L_{\lambda_1,\lambda_2}{}^{\alpha}(m-n;b^2)=\int_0^\infty qdq\,H_{\lambda_1,\lambda_2}{}^{\alpha}(m-n;q^2)J_{\lambda_2-\lambda_1}(bq).\quad(5.3)$$

It should be noted that in Eq. (5.2), there is no summation over intermediate spin directions. The sum rules are true for every k. We obtain the result of Coester and Roepstorff[19] that if the form factors are to have a nontrivial q^2 dependence, then saturation will be obtained only with an infinite set of states with unbounded spin. This may be seen by taking k as large as we please in Eq. (5.2). It is hoped that approximate saturations may be obtained for low values of k.

We close this section with some further relations between L and H. With the assumption that $H_{\lambda_1,\lambda_2}(q^2)/q^{|\lambda_1-\lambda_2|}$ satisfies an unsubtracted dispersion relation, Eq. (4.14), we find

$$L_{\lambda_1,\lambda_2}(b)=\frac{1}{\pi}\int q^{|\lambda_2-\lambda_1|}K_{\lambda_2-\lambda_1}(qb)\,\mathrm{Im}\left(\frac{H_{\lambda_1,\lambda_2}(q^2)}{q^{|\lambda_2-\lambda_1|}}\right)dq^2,\quad(5.4)$$

with the integration contour being the same as that of (4.14). This relation gives information on the asymptotic behavior of L in terms of the masses contributing to the dispersive part of H. The amplitude $H_{\lambda_1,\lambda_2}(q^2)$ may be recovered from $L_{\lambda_1,\lambda_2}(b)$ by

$$H_{\lambda_1,\lambda_2}(q^2)=\int_0^\infty bdb\,J_{\lambda_2-\lambda_1}(qb)\,L_{\lambda_1,\lambda_2}(b).\quad(5.5)$$

APPENDIX: SCALAR VERTEX FUNCTION

For completeness, we state some results about the parametrization and singularity structure of the vertex function for scalar amplitudes. If

$$\Delta_{\lambda_1,\lambda_2}(p_1,p_2)=(2\pi)^3(2\omega_1\omega_2)^{1/2}\langle p_1,\lambda_1|J(0)|p_2,\lambda_2\rangle,\quad(\mathrm{A}1)$$

with J being a scalar current, then in the Breit system

$$\Delta_{\lambda_1,\lambda_2}(q^2)=(-1)^{j_1-\lambda_1}\left(\frac{m_1^2+m_2^2+q^2+T}{2m_1m_2}\right)^{\lambda_2}\times\sum_J\langle J,0|j_1,-\lambda_1;j_2,\lambda_2\rangle T^J S_J,\quad(\mathrm{A}2)$$

where S_J has no kinematic singularities. If J is the divergence of the vector current J^μ, then

$$S_J=(m_2^2-m_1^2)A_J-q^2B_J-J^{1/2}C_J.\quad(\mathrm{A}3)$$

Taking over some results from the text, we may show that

$$d_{\lambda_1,\eta_1}{}^{-1(j_1)}(\psi_1)\Delta_{\eta_1,\eta_2}(q^2)d_{\eta_2,\lambda_2}{}^{(j_2)}(\psi_2)/q^{|\lambda_2-\lambda_1|}$$

has no kinematic singularities. ψ_1 and ψ_2 are defined in Eq. (3.3).

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[17] For the case where the external states are nucleons $H_{1/2,1/2}\sim F_1$ and $H_{1/2-1/2}/q\sim F_2$ for which there seems to be evidence that they satisfy unsubtracted dispersion relations. R. Taylor, in Proceedings of the 1967 International Symposium on Electrons and Photon Interactions at High Energies, Stanford, 1967 (unpublished).

[18] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Co., New York, 1954), Vol. II.

[19] F. Coester and G. Roepstorff, Phys. Rev. **155**, B1583 (1967).